CYCLIC SIEVING FOR TORSION PAIRS IN THE CLUSTER CATEGORY OF DYNKIN TYPE A_n

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Abstract. Recently, a combinatorial model for torsion pairs in the cluster category of Dynkin type A_n was introduced, and used to derive an explicit formula for their number. In this article we determine the number of torsion pairs that are invariant under b-fold application of Auslander-Reiten translation.

It turns out that the set of torsion pairs together with Auslander-Reiten translation, and a natural q-analogue of the formula for the number of all torsion pairs exhibits the cyclic sieving phenomenon.

1. Introduction

1.1. **Torsion Pairs and Ptolemy Diagrams.** Very recently, Ptolemy diagrams were introduced by Thorsten Holm, Peter Jørgensen and Martin Rubey in [3] as a combinatorial model for torsion pairs in the cluster category of Dynkin type A_n. Similar to triangulations of the $(n+3)$-gon, which can be regarded as a combinatorial model for tilting objects, Ptolemy diagrams are certain subsets of the set of (proper) diagonals of an $(n+3)$-gon with a distinguished base edge. As in the case of triangulations, each such diagonal corresponds to an indecomposable object in the cluster category.

The set $\mathcal{P}$ of Ptolemy diagrams with distinguished base edge can be described recursively as indicated in Figure 1, see [3, Proposition 2.4]. More precisely, $\mathcal{P}$ is the disjoint union of

(i) the degenerate Ptolemy diagram, consisting of two vertices and the distinguished base edge only,
(ii) a triangle with a distinguished base edge and two Ptolemy diagrams glued along their distinguished base edges onto the other edges,
(iii) a clique, *i.e.*, a diagram with at least four edges and all diagonals present, with a distinguished base edge and Ptolemy diagrams glued along their distinguished base edges onto the other edges.

Key words and phrases. cluster category, torsion pair, Ptolemy diagram, polygon dissection, cyclic sieving.

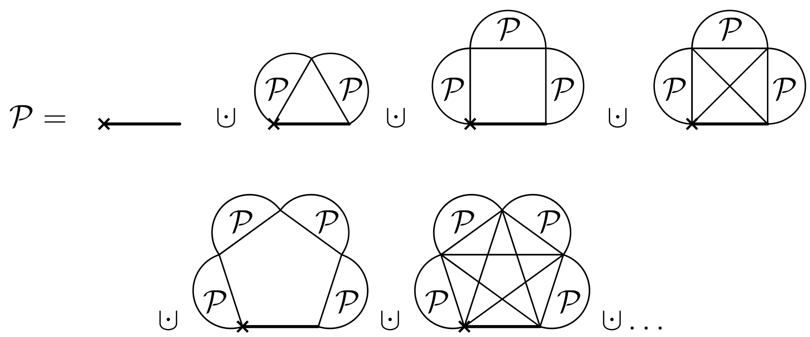


Figure 1. The decomposition of the set of Ptolemy diagrams with a distinguished base edge.

(iv) an empty cell, *i.e.*, a polygon with at least four edges without diagonals, with a distinguished base edge and Ptolemy diagrams glued along their distinguished base edges onto the other edges,

In other words, Ptolemy diagrams are obtained by gluing together 'elementary' Ptolemy diagrams, *i.e.*, triangles, cliques and empty cells. Thus, one could also describe Ptolemy diagrams as polygon dissections (see, *e.g.* [6, Proposition 6.2.1 (vi)]), in which each region of size at least four receives one of two colours.

A natural operation on Ptolemy diagrams is (counterclockwise) rotation. In the cluster category, rotation corresponds to Auslander-Reiten translation τ, or, equivalently, application of the suspension functor Σ. Both the total number of Ptolemy diagrams with distinguished base edge, and the number of diagrams up to rotation, were already determined, see [3, Theorem B and Proposition 3.4]. The central goal of this article is to determine the number of Ptolemy diagrams that are invariant under rotation by a given angle.

Apart from rotation, another natural operation on Ptolemy diagrams consists of replacing every clique by an empty cell and vice versa. More generally, given *any* set of diagonals $\mathfrak{A}$ of the $(n+3)$-gon, let $\operatorname{nc}\mathfrak{A}$ be the set of diagonals that cross no diagonal in $\mathfrak{A}$. It turns out that $\mathfrak{A}$ is a Ptolemy diagram if and only if $\mathfrak{A} = \operatorname{nc}\operatorname{nc}\mathfrak{A}$, see [3, Proposition 2.6]. Note that precisely those Ptolemy diagrams that are triangulations remain invariant under this operation.

Let A be a subcategory of the cluster category of Dynkin type A_n, closed under direct sums and direct summands, that corresponds to a set of diagonals $\mathfrak{A}$. Then the *perpendicular subcategory* $\mathsf{A}^\perp$ is the subcategory of the same cluster category that consists of those objects

that have only the zero map to any object in A:

$$\mathsf{A}^{\perp} = \{c : Hom(c, a) = 0 \text{ for all } a \in \mathsf{A}\}$$

One can show that $(\mathsf{A}, \mathsf{A}^{\perp})$ is a torsion pair if and only if $\mathfrak{A}$ is a Ptolemy diagram, see [3, Proposition 2.3]. In this setting, $\mathsf{A}^{\perp}$ also corresponds to a Ptolemy diagram, namely $\Sigma\,\mathrm{nc}\,\mathfrak{A}$. As a corollary of our main result we also obtain the number of Ptolemy diagrams invariant under taking perpendicular subcategories a given number of times.

1.2. **Cyclic Sieving Phenomena.** The cyclic sieving phenomenon was first described in 2004 by Victor Reiner, Denis Stanton and Dennis White [4]. It involves a finite set $\mathcal{X}$, a cyclic group C of order n acting on $\mathcal{X}$, and a polynomial $X(q)$.

Definition 1.1. The triple $\big(\mathcal{X}, C, X(q)\big)$ *exhibits the cyclic sieving phenomenon* if for every $c \in C$ we have

$$X(\omega_{o(c)}) = |\mathcal{X}^c|,$$

where $o(c)$ denotes the order of $c \in C$, ω_d is a d^{th} primitive root of unity and $\mathcal{X}^c = \{x \in \mathcal{X} : c(x) = x\}$ denotes the set of fixed points of $\mathcal{X}$ under the action of $c \in C$.

In particular, $X(1) = |\mathcal{X}|$, *i.e.*, $X(q)$ is a q-analogue of the generating function for $\mathcal{X}$.

We remark that the cyclic sieving polynomial X is unique only modulo $q^n - 1$. However, for the unique cyclic sieving polynomial of degree at most $n - 1$, there is an alternative description, which makes the combinatorics of the orbit structure of C acting on $\mathcal{X}$ explicit.

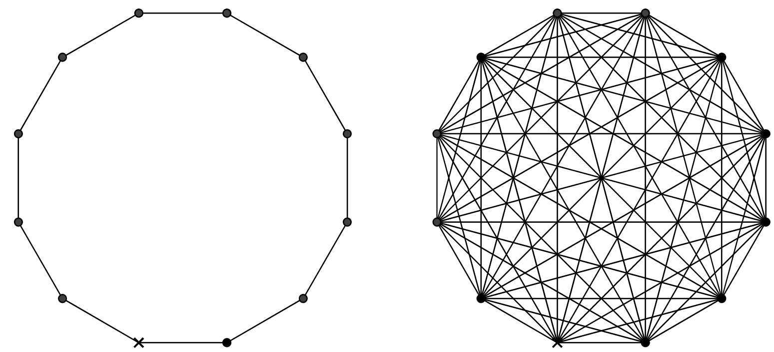
Proposition 1.2 ([4, Proposition 2.1])**.** *Let the* stabiliser order *of a C-orbit of $\mathcal{X}$ be the number of elements $c \in C$ that fix an element (and therefore all elements) in this orbit.*

For $0 \leq \ell < n$ let a_ℓ be the number of orbits of the action of C on $\mathcal{X}$, whose stabiliser order divides ℓ, and let $X(q) = \sum_{\ell=0}^{n-1} a_\ell q^\ell$. Then the triple $\big(\mathcal{X}, C, X(q)\big)$ exhibits the cyclic sieving phenomenon.

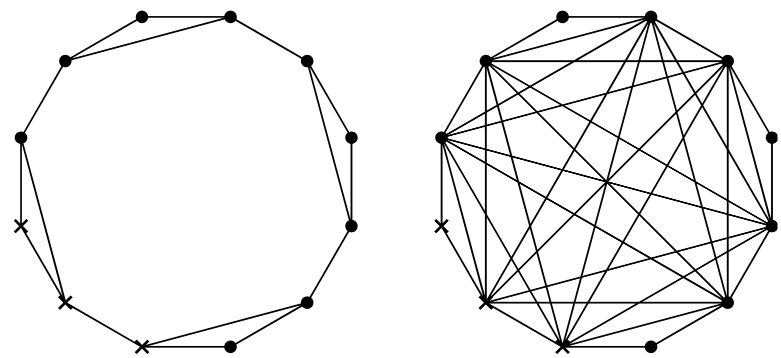
In particular, a_0 is the total number of orbits and a_1 is the number of free orbits.

1.3. **Main Theorems.**

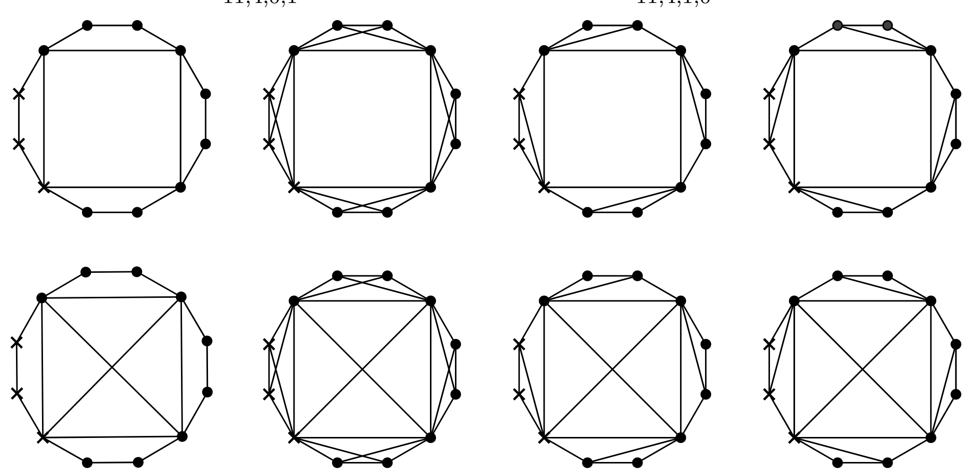
Theorem 1.3. *Let $\mathcal{P}_{N,k,\ell,m}$ be the set of Ptolemy diagrams on the $(N+1)$-gon with a distinguished base edge, with k triangles, ℓ cliques of size*



$P^{(4)}_{11,0,0,1} = 1$ $\qquad$ $P^{(4)}_{11,0,1,0} = 1$



$P^{(4)}_{11,4,0,1} = 3$ $\qquad$ $P^{(4)}_{11,4,1,0} = 3$



$P^{(4)}_{11,0,0,5} = P^{(4)}_{11,0,4,1} = 3$ $\qquad$ $P^{(4)}_{11,8,0,1} = 6$

$P^{(4)}_{11,0,1,4} = P^{(4)}_{11,0,5,0} = 3$ $\qquad$ $P^{(4)}_{11,8,1,0} = 6$

FIGURE 2. Ptolemy diagrams on the twelve-gon with fourfold symmetry. Crosses indicate different choices of base vertices.

at least four and m empty cells of size at least four. Then the cardinality of $\mathcal{P}_{N,k,\ell,m}$ is

$$P_{N,k,\ell,m} = \frac{1}{N}\binom{N-1+k+\ell+m}{N-1,k,\ell,m}\binom{N-2-k-\ell-m}{\ell+m-1},$$

where we set $\binom{n}{n}=1$ for $n\in\mathbb{Z}$.

Theorem 1.4. *Let $P^{(d)}_{N,k,\ell,m}$ be the number of Ptolemy diagrams in $\mathcal{P}_{N,k,\ell,m}$ that are invariant under rotation by $2\pi/d$. Then, for $d\geq 2$ a divisor of $N+1$,*

$$P^{(d)}_{N,k,\ell,m} = \binom{\frac{N+1}{d}-1+\lfloor\frac{k+\ell+m}{d}\rfloor}{\frac{N+1}{d}-1,\lfloor\frac{k}{d}\rfloor,\lfloor\frac{\ell}{d}\rfloor,\lfloor\frac{m}{d}\rfloor}\binom{\lfloor\frac{N-2-k-\ell-m}{d}\rfloor}{\lfloor\frac{\ell+m-1}{d}\rfloor}$$

if $N-2-k-\ell-m\geq \ell+m-1$ and

(i) $d=2$ and $k\equiv\ell\equiv m\equiv 0 \pmod d$, or
(ii) $d=3$ and $k\equiv 1 \pmod d$, $\ell\equiv m\equiv 0 \pmod d$, or
(iii) $d\geq 2$ arbitrary and $k\equiv\ell\equiv m-1\equiv 0 \pmod d$ or $k\equiv\ell-1\equiv m\equiv 0 \pmod d$.

In all other cases, $P^{(d)}_{N,k,\ell,m}=0$.

Remark 1. Since $P^{(\frac{N+1}{b})}_{N,k,\ell,m} = P^{(\frac{N+1}{g})}_{N,k,\ell,m}$, where g is the greatest common divisor of $N+1$ and b, the assumption that d is an integer is not a real restriction.

Remark 2. Expressions for the generating functions of $P^{(d)}_{N,k,\ell,m}$ are given in Lemma 2.2.

As an illustration, the Ptolemy diagrams on the twelve-gon that are invariant under rotation by $\pi/2$ are shown in Figure 2.

Since $\mathfrak{A}^{\perp}=\Sigma\,\mathrm{nc}\,\mathfrak{A}=\tau\,\mathrm{nc}\,\mathfrak{A}$, this theorem also determines the number of Ptolemy diagrams whose corresponding subcategory is invariant under taking perpendicular subcategories b times:

Corollary 1.5. *Let $P^{\perp^b}_{N,k,\ell,m}$ be the number of Ptolemy diagrams in $\mathcal{P}_{N,k,\ell,m}$ invariant under b-fold application of taking perpendiculars. Let $d=\frac{N+1}{b}$, then*

$$P^{\perp^b}_{N,k,\ell,m} = 2^e\binom{\frac{N+1}{2}-1+\frac{k}{2}+e}{\frac{N+1}{2}-1,\frac{k}{2},e}\binom{\frac{N+1}{2}-2-\frac{k}{2}-e}{e-1},$$

if b is odd, $d=2$, $k\equiv 0 \pmod d$ and $\ell=m=e$, in which case the the central region is degenerate, and

$$P^{\perp^b}_{N,k,\ell,m} = P^{(d)}_{N,k,\ell,m},$$

if b is odd, $d = 3$, $k \equiv 1 \pmod{d}$ and $\ell = m = 0$, in which case the Ptolemy diagram is a triangulation, or if b is even. In all other cases, there are no such Ptolemy diagrams.

As an illustration, the Ptolemy diagrams on the hexagon that are invariant under taking perpendiculars three times are shown in Figure 3.

Proof. If b is even we have $\mathfrak{A}^{\perp^b} = (\tau\,\mathrm{nc})^b\mathfrak{A} = \tau^b\mathfrak{A}$, since nc is an involution on Ptolemy diagrams.

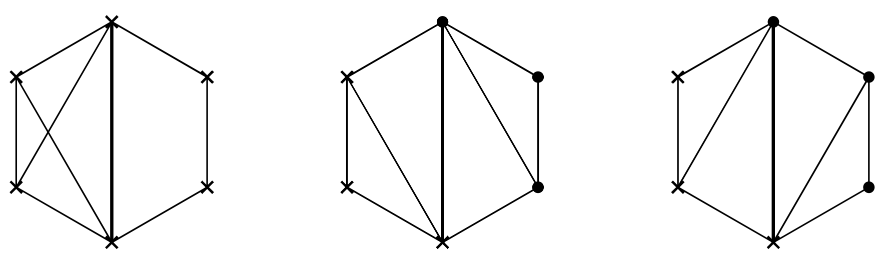
If b is odd, $\mathfrak{A}^{\perp^b} = \tau^b\,\mathrm{nc}\,\mathfrak{A}$. Thus the central region must be degenerate or a triangle. If it is degenerate, we can glue any Ptolemy diagram $\mathfrak{A}$ on one side, and $\mathrm{nc}\,\mathfrak{A}$ on the other side of the edge, to obtain an invariant diagram. Thus

$$\begin{aligned} P^{\perp^b}_{N,k,e,e} &= \frac{N+1}{2}\sum_{\ell=0}^{e} P_{\frac{N+1}{2},\frac{k}{2},\ell,e-\ell} \\ &= \sum_{\ell=0}^{e} \binom{\frac{N+1}{2}-1+\frac{k}{2}+e}{\frac{N+1}{2}-1,\frac{k}{2},e}\binom{e}{\ell}\binom{\frac{N+1}{2}-2-\frac{k}{2}-e}{e-1}, \end{aligned}$$

as claimed.

Otherwise, we have to glue three Ptolemy diagrams $\mathfrak{A}_1$, $\mathfrak{A}_2$ and $\mathfrak{A}_3$ onto the triangle, one on each side. It follows that $\mathfrak{A}_1 = \mathrm{nc}\,\mathfrak{A}_2 = \mathrm{nc}^2\,\mathfrak{A}_3 = \mathrm{nc}^3\,\mathfrak{A}_1$, and thus that $\mathfrak{A}_1 = \mathfrak{A}_2 = \mathfrak{A}_3$ is a triangulation. $\square$

As another corollary we obtain a (relatively) explicit expression for the number of Ptolemy diagrams up to rotation. A (relatively complicated) expression for the corresponding generating function was already given in [3, Proposition 3.4].



$P^{\perp^3}_{5,0,1,1} = 6$ $P^{\perp^3}_{5,4,0,0} = 6$

FIGURE 3. Ptolemy diagrams on the hexagon invariant under taking perpendiculars three times. Crosses indicate different choices of base vertices.

Corollary 1.6. *The number of Ptolemy diagrams in $\mathcal{P}_{N,k,\ell,m}$ up to rotation is*

$$\frac{1}{N+1}\sum_{d|N+1}\phi(d)P^{(d)}_{N,k,\ell,m},$$

where we set $P^{(1)}_{N,k,\ell,m} = P_{N,k,\ell,m}$.

Note however that this is not as explicit as it may seem, since summands vanish depending on the congruence class modulo d.

Proof. For every $d|N+1$ there are $\phi(d)$ elements of order d in the cyclic group of rotations of the $(N+1)-gon$, and for each of these rotations there are $P^{(d)}_{N,k,\ell,m}$ Ptolemy diagrams that are invariant. The corollary now follows from the Cauchy-Frobenius formula for the number of orbits

$$\frac{1}{N+1}\sum_{b=0}^{N}\#\{\text{Ptolemy diagrams fixed by } \tau^b\}.$$

□

The remainder of this section is dedicated to a rephrasing of Theorems 1.3 and 1.4 as a cyclic sieving phenomenon.

Definition 1.7. For $0 \leq k \leq n$ the *q-binomial coefficient* is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!},$$

where $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$ and $[n]_q = 1+q\cdots+q^{n-1}$. Analogously, the *q-multinomial coefficient* is

$$\begin{bmatrix} n_1+n_2+\dots n_\ell \\ n_1,n_2,\dots,n_\ell \end{bmatrix}_q = \frac{[n_1+n_2+\dots n_\ell]_q!}{[n_1]_q![n_2]_q!\dots[n_\ell]_q!},$$

where $n_1, n_2, \dots n_\ell$ are non-negative integers.

Probably *the* reason why cyclic sieving is called a *phenomenon*, is that the cyclic sieving polynomial $X(q)$ is frequently obtained from the ordinary counting function merely by replacing binomials with q-binomials and the like. This is also the case in our situation:

Theorem 1.8. *Let $\mathcal{P}_{N,k,\ell,m}$ be the set of Ptolemy diagrams on the $(N+1)$-gon, with k triangles, ℓ cliques of size at least four and m empty cells of size at least four. Let τ be the operation of rotation acting on this set, and let*

$$P_{N,k,\ell,m}(q) = \frac{1}{[N]_q}\begin{bmatrix} N-1+k+\ell+m \\ N-1,k,\ell,m \end{bmatrix}_q \begin{bmatrix} N-2-k-\ell-m \\ \ell+m-1 \end{bmatrix}_q,$$

where we set $\begin{bmatrix} n \\ n \end{bmatrix}_q = 1$ *for* $n \in \mathbb{Z}$.

Then $\big(\mathcal{P}_{N,k,\ell,m}, \langle\tau\rangle, P_{N,k,\ell,m}(q)\big)$ *exhibits the cyclic sieving phenomenon.*

Remark 3. Putting $\ell = 0$ in the theorem above we obtain polygon dissections with a given number of regions. By contrast, Victor Reiner, Denis Stanton and Dennis White [4] showed that polygon dissections with a given number of diagonals also exhibit the cyclic sieving phenomenon. However, so far we did not manage to find a common generalisation, not even for the formula giving the total number of dissections of the n-gon with k diagonals, *i.e.*

$$\frac{1}{n+k}\binom{n+k}{k+1}\binom{n-3}{k}$$

and the formula in Theorem 1.3.

A collection of (partially conjectural) instances of the cyclic sieving phenomenon that involve various subsets of (non-crossing) diagonals of the n-gon was given by Alan Guo in [2]. Two common features of these and also the one described in the present article are that the cyclic sieving polynomial is a simple product of q-binomial coefficients, and to date, no representation theoretic proof along the lines of [4, Lemma 2.4] is available...

2. Counting Ptolemy diagrams

In this section we determine the total number of Ptolemy diagrams as well as the number of Ptolemy diagrams invariant under rotation by a given angle. That is, we provide proofs of Theorem 1.3 and Theorem 1.4.

We do so by exhibiting equations for the generating functions, whose coefficients we then extract using Lagrange inversion:

Theorem 2.1 (Lagrange inversion)**.** *Let* $F(z)$ *be a formal power series with* $[z^0]F(z) = 0$ *and* $[z^1]F(z) \neq 0$. *Let* $F^{(-1)}(z)$ *its compositional inverse and* $H(z)$ *an arbitrary formal power series. Then the coefficient of* z^n *in* $H(F^{(-1)}(z))$ *is*

$$[z^n]H(F^{(-1)}(z)) = \frac{1}{n}[z^{n-1}]H'(z)\left(\frac{F(z)}{z}\right)^{-n}.$$

Proof. A proof may be found, for example, in [6, Corollary 5.4.3]. □

Proof of Theorem 1.3. Let $P_{N,k,\ell,m}$ be the number of Ptolemy diagrams on the $(N+1)$-gon with k triangles, ℓ cliques of size at least four and m

empty cells of size at least four. Then the ordinary generating function for Ptolemy diagrams is

$$\mathcal{P}(z) = \mathcal{P}(z, x, y_1, y_2) = \sum_{N\geq 1, k,\ell,m\geq 0} P_{N,k,\ell,m}\; z^N x^k y_1^\ell y_2^m.$$

Translating the recursive description for the set of Ptolemy diagrams given in the introduction into an equation for their generating function we obtain

$$\mathcal{P}(z) = z + x\mathcal{P}(z)^2 + (y_1 + y_2)\frac{\mathcal{P}(z)^3}{1 - P(z)},$$

or equivalently,

$$\mathcal{P}(z)\left(1 - x\mathcal{P}(z) - (y_1 + y_2)\frac{\mathcal{P}(z)^2}{1 - \mathcal{P}(z)}\right) = z.$$

We are now able to apply Lagrange inversion to obtain formulae for the coefficients of $\mathcal{P}(z)$: setting $Q(z) = z\left(1 - xz - (y_1 + y_2)\frac{z^2}{1-z}\right)$ we have $z = Q(\mathcal{P})$, *i.e.* Q is the compositional inverse of $\mathcal{P}$. Therefore

$$[z^N]\mathcal{P}(z) = \frac{1}{N}[z^{N-1}]\left(\frac{Q(z)}{z}\right)^{-N}.$$

Applying the multinomial theorem

$$(1 - x - y_1 - y_2)^{-N} = \sum_{k,\ell,m} \binom{N - 1 + k + \ell + m}{N - 1, k, \ell, m} x^k y_1^\ell y_2^m$$

we find

$$\begin{aligned}
\left(\frac{Q(z)}{z}\right)^{-N} &= \left(1 - xz - (y_1 + y_2)\frac{z^2}{1 - z}\right)^{-N}\\
&= \sum_{k,\ell,m} \binom{N - 1 + k + \ell + m}{N - 1, k, \ell, m}(xz)^k y_1^\ell y_2^m \frac{z^{2(\ell+m)}}{(1 - z)^{\ell+m}}\\
&= \sum_{k,\ell,m} \binom{N - 1 + k + \ell + m}{N - 1, k, \ell, m} x^k y_1^\ell y_2^m z^{k+2(\ell+m)}\\
&\qquad \sum_i \binom{\ell + m - 1 + i}{\ell + m - 1} z^i.
\end{aligned}$$

Extracting the coefficient of z^{N-1} by setting $i = N-1-k-2(\ell+m)$ we obtain the desired formula

$$\begin{aligned} P_{N,k,\ell,m} &= [z^N x^k y_1^\ell y_2^m]\mathcal{P}(z,x,y_1,y_2) \\ &= \frac{1}{N}\binom{N-1+k+\ell+m}{N-1,k,\ell,m}\binom{N-k-\ell-m-2}{\ell+m-1}. \end{aligned}$$

□

Lemma 2.2. *Let*

$$\mathcal{P}(z,x,y_1,y_2) = \sum_{N\geq 1,k,\ell,m\geq 0} P_{N,k,\ell,m}\; z^N x^k y_1^\ell y_2^m$$

be the generating function for Ptolemy diagrams. Then the generating function for Ptolemy diagrams that are invariant under rotation by $2\pi/d$ *equals*

$$\begin{aligned} &\frac{1}{z}\bar z\,\mathcal{P}'(\bar z,\bar x,\bar y_1,\bar y_2)\left(1+\mathcal{P}(\bar z,\bar x,\bar y_1,\bar y_2)\frac{y_1+y_2}{1-\mathcal{P}(\bar z,\bar x,\bar y_1,\bar y_2)}\right) && \text{for } d=2,\\ &\frac{1}{z}\bar z\,\mathcal{P}'(\bar z,\bar x,\bar y_1,\bar y_2)\left(x+\mathcal{P}(\bar z,\bar x,\bar y_1,\bar y_2)\frac{y_1+y_2}{1-\mathcal{P}(\bar z,\bar x,\bar y_1,\bar y_2)}\right) && \text{for } d=3,\\ &\frac{1}{z}\bar z\,\mathcal{P}'(\bar z,\bar x,\bar y_1,\bar y_2)\frac{y_1+y_2}{1-\mathcal{P}(\bar z,\bar x,\bar y_1,\bar y_2)} && \text{for } d\geq 4. \end{aligned}$$

In these formulae, we set $\bar z = z^d$, $\bar x = x^d$, $\bar y_1 = y_1^d$ *and* $\bar y_2 = y_2^d$, *and the derivative is with respect to* $\bar z$.

Proof. Up to this point we always distinguished an edge of the $(N+1)$-gon when counting Ptolemy diagrams. Clearly, this is equivalent to marking one of the $N+1$ vertices of the polygon. For the recursive description in the introduction the former seemed more natural, but in this proof it will be more convenient to mark a vertex, which we will call the *distinguished base vertex* henceforth.

For $d \geq 2$ we can construct a Ptolemy diagram invariant under rotation by $2\pi/d$ as follows: for any multiple s of d, we choose a list of s/d Ptolemy diagrams. In the first of these, we select one vertex other than the distinguished base vertex, which will become the distinguished base vertex of the diagram we are about to construct. Then we glue the Ptolemy diagrams of d identical copies of this list in order along their distinguished base edges onto the edges of a polygon with s vertices. Of course, in the degenerate case $s = 2$ we simply have two identical Ptolemy diagrams which we glue onto each other along their distinguished base edges. Finally, if $s \geq 4$, we choose whether this central region should be an clique or an empty cell.

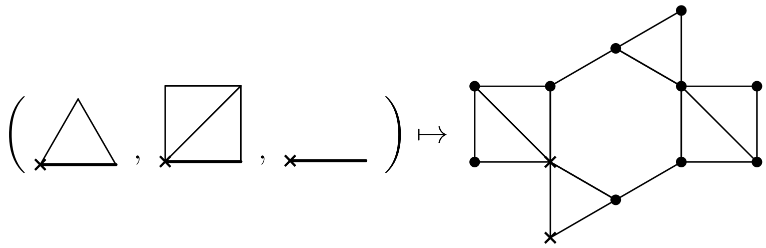


FIGURE 4. Constructing a Ptolemy diagram invariant under rotation by $\pi/4$. Crosses indicate different choices of base vertices.

Conversely, given a Ptolemy diagram invariant under rotation by $2\pi/d$, the central region is the region (or possibly the diameter) containing the geometric center of the polygon when drawn regular and with all diagonals straight. Let s be the number of vertices of this central region. Cutting out the central region we obtain a circular arrangement of smaller Ptolemy diagrams. We now select the first s/d diagrams in this arrangement, starting with the one that contains the distinguished base edge, *i.e.*, the edge that comes just before the distinguished base vertex when going clockwise. An example for this correspondence with $d = 2$ and $s = 6$ can be found in Figure 4.

Let us translate this description into the expressions for the generating functions as given in the statement of the lemma. To this end, recall that the generating function of *pointed* Ptolemy diagrams, *i.e.* diagrams with a vertex other than the distinguished base vertex selected, equals $z\mathcal{P}'(z, x, y_1, y_2)$ (see for example [1, Section 2.1]), and the generating function of *lists* of Ptolemy diagrams is $1/\left(1 - \mathcal{P}(z, x, y_1, y_2)\right)$. Since we attach every Ptolemy diagram in the list d times, the number of vertices, triangles, etc., in each diagram has to be multiplied by d, which is accomplished by replacing z by $\bar{z}$, t by $\bar{x}$, etc. Finally, we have to divide by z, because this variable marks the number of vertices *minus one*. □

Proof of Theorem 1.4. Following Lemma 2.2, we will treat $d = 2$, $d = 3$ and $d \geq 4$ separately. However, let us first compute the expansions of $\bar{z}\mathcal{P}'(\bar{z}, \bar{x}, \bar{y}_1, \bar{y}_2)$ and $\bar{z}\frac{\mathcal{P}'(\bar{z},\bar{x},\bar{y}_1,\bar{y}_2)}{1-\mathcal{P}(\bar{z},\bar{x},\bar{y}_1,\bar{y}_2)}$, which will be needed for all three

cases. The first expansion is essentially Theorem 1.3:

$$\begin{aligned}\bar z\mathcal{P}'(\bar z,\bar x,\bar y_1,\bar y_2) &= \sum_{n,k,\ell,m} nP_{n,k,\ell,m}\bar z^n\bar x^k\bar y_1^\ell\bar y_2^m\\ &= \sum_{n,k,\ell,m}\binom{n-1+k+\ell+m}{n-1,k,\ell,m}\binom{n-2-k-\ell-m}{\ell+m-1}\bar z^n\bar x^k\bar y_1^\ell\bar y_2^m.\end{aligned}$$

For the second, we compute

$$\begin{aligned}[\bar z^n]\bar z\frac{\mathcal{P}'(\bar z,\bar x,\bar y_1,\bar y_2)}{1-\mathcal{P}(\bar z,\bar x,\bar y_1,\bar y_2)} &= [\bar z^{n-1}]\frac{\mathcal{P}'(\bar z,\bar x,\bar y_1,\bar y_2)}{1-\mathcal{P}(\bar z,\bar x,\bar y_1,\bar y_2)}\\ &= [\bar z^{n-1}]\left(\log\frac{1}{1-\mathcal{P}(\bar z,\bar x,\bar y_1,\bar y_2)}\right)'\\ &= n[\bar z^n]\log\frac{1}{1-\mathcal{P}(\bar z,\bar x,\bar y_1,\bar y_2)}\\ &= [\bar z^{n-1}]\frac{1}{1-\bar z}\left(\frac{Q(\bar z)}{\bar z}\right)^{-n}.\end{aligned}$$

In the last line we used Lagrange inversion with $H(\bar z)=\log\left(1/(1-\bar z)\right)$ and $Q(\bar z)=\bar z\left(1-\bar x\bar z-(\bar y_1+\bar y_2)\frac{\bar z^2}{1-\bar z}\right)$. The expansion of $\left(\frac{Q(\bar z)}{\bar z}\right)^{-n}$ was already computed in the proof of Theorem 1.3; taking into account the additional factor $\frac{1}{1-\bar z}$ we obtain

$$\bar z\frac{\mathcal{P}'(\bar z)}{1-\mathcal{P}(\bar z)} = \sum_{n,k,\ell,m}\binom{n-1+k+\ell+m}{n-1,k,\ell,m}\binom{n-1-k-\ell-m}{\ell+m}\bar z^n\bar x^k\bar y_1^\ell\bar y_2^m.$$

Case $d\geq 4$**.** By Lemma 2.2, we need to compute the coefficient of $z^N=z^{kn-1}=\frac{1}{z}\bar z^n$ in

$$\frac{1}{z}\bar z\,\mathcal{P}'(\bar z,\bar x,\bar y_1,\bar y_2)\frac{y_1+y_2}{1-\mathcal{P}(\bar z,\bar x,\bar y_1,\bar y_2)},$$

where $\bar z=z^d$, $\bar x=x^d$, $\bar y_1=y_1^d$ and $\bar y_2=y_2^d$.

Thus, the exponent of x and of one of y_1 and y_2 must be divisible by d, while the exponent of the other variable equals 1 (mod d). We conclude that the number $P^{(d)}_{N,k,\ell,m}$ of Ptolemy diagrams in $\mathcal{P}_{N,k,\ell,m}$ that are invariant under rotation by $\frac{2\pi}{d}$ is

$$\binom{\frac{N+1}{d}-1+\frac{k}{d}+\frac{\ell}{d}+\frac{m-1}{d}}{\frac{N+1}{d}-1,\frac{k}{d},\frac{\ell}{d},\frac{m-1}{d}}\binom{\frac{N+1}{d}-1-\frac{k}{d}-\frac{\ell}{d}-\frac{m-1}{d}}{\frac{\ell}{d}+\frac{m-1}{d}}$$

if $N+1 \equiv k \equiv \ell \equiv m-1 \equiv 0 \pmod{d}$,

$$\binom{\frac{N+1}{d} - 1 + \frac{k}{d} + \frac{\ell-1}{d} + \frac{m}{d}}{\frac{N+1}{d} - 1, \frac{k}{d}, \frac{\ell-1}{d}, \frac{m}{d}} \binom{\frac{N+1}{d} - 1 - \frac{k}{d} - \frac{\ell-1}{d} - \frac{m}{d}}{\frac{\ell-1}{d} + \frac{m}{d}}$$

if $N+1 \equiv k \equiv \ell - 1 \equiv m \equiv 0 \pmod{d}$, and 0 otherwise, as claimed.

Case $d=2$**.** By Lemma 2.2, we need to compute the coefficient of z^N in

$$\frac{1}{z}\bar{z}\,\mathcal{P}'(\bar{z},\bar{x},\bar{y}_1,\bar{y}_2)\left(1+\mathcal{P}(\bar{z},\bar{x},\bar{y}_1,\bar{y}_2)\frac{y_1+y_2}{1-\mathcal{P}(\bar{z},\bar{x},\bar{y}_1,\bar{y}_2)}\right)$$

$$= \frac{1}{z}\bar{z}\,\mathcal{P}'(\bar{z},\bar{x},\bar{y}_1,\bar{y}_2) \tag{1}$$

$$+ \frac{1}{z}(y_1+y_2)\left(\bar{z}\frac{\mathcal{P}'(\bar{z},\bar{x},\bar{y}_1,\bar{y}_2)}{1-P(\bar{z},\bar{x},\bar{y}_1,\bar{y}_2)} - \bar{z}\mathcal{P}'(\bar{z},\bar{x},\bar{y}_1,\bar{y}_2)\right) \tag{2}$$

Using the expansions for $\bar{z}\mathcal{P}'(\bar{z})$ and $\bar{z}\frac{\mathcal{P}'(\bar{z})}{1-\mathcal{P}(\bar{z})}$ and the recurrence $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ we obtain

$$\begin{aligned}
&\bar{z}\frac{\mathcal{P}'(\bar{z})}{1-\mathcal{P}(\bar{z})} - \bar{z}\mathcal{P}'(\bar{z}) \\
&\quad = \sum_{n,k,\ell,m} \binom{n-1+k+\ell+m}{n-1,k,\ell,m}\binom{n-1-k-\ell-m}{\ell+m} \bar{z}^n\bar{x}^k\bar{y}_1^\ell\bar{y}_2^m \\
&\qquad - \sum_{n,k,\ell,m} \binom{n-1+k+\ell+m}{n-1,k,\ell,m}\binom{n-2-k-\ell-m}{\ell+m-1} \bar{z}^n\bar{x}^k\bar{y}_1^\ell\bar{y}_2^m \\
&\quad = \sum_{n,k,\ell,m} \binom{n-1+k+\ell+m}{n-1,k,\ell,m}\binom{n-2-k-\ell-m}{\ell+m} \bar{z}^n\bar{x}^k\bar{y}_1^\ell\bar{y}_2^m.
\end{aligned}$$

We can now extract the coefficient of z^N separately from the summands (1) and (2) to derive the expressions claimed.

Case $d=3$**.** By Lemma 2.2, we need to compute the coefficient of y^N in

$$\frac{1}{z}\bar{z}\,\mathcal{P}'(\bar{z},\bar{x},\bar{y}_1,\bar{y}_2)\left(x+\mathcal{P}(\bar{z},\bar{x},\bar{y}_1,\bar{y}_2)\frac{y_1+y_2}{1-\mathcal{P}(\bar{z},\bar{x},\bar{y}_1,\bar{y}_2)}\right)$$

$$= \frac{1}{z}x\bar{z}\mathcal{P}'(\bar{z},\bar{x},\bar{y}_1,\bar{y}_2) \tag{3}$$

$$+ \frac{1}{z}(y_1+y_2)\left(\bar{z}\frac{\mathcal{P}'(\bar{z},\bar{x},\bar{y}_1,\bar{y}_2)}{1-P(\bar{z},\bar{x},\bar{y}_1,\bar{y}_2)} - \bar{z}\mathcal{P}'(\bar{z},\bar{x},\bar{y}_1,\bar{y}_2)\right). \tag{4}$$

This is completely analogous to the case $d=2$, the only difference being the factor x in the first summand. $\square$

3. Evaluating the q-binomial coefficients

In this section we show Theorem 1.8 by evaluating the expression given there at roots of unity, and thus checking that the result indeed equals the expression in Theorem 1.4. The evaluations of the q-binomial coefficients will be based on the q-Lucas theorem:

Lemma 3.1 (q-Lucas theorem)**.** *Let ω be a primitive d^{th} root of unity and a and b non-negative integers. Then*

$$\begin{bmatrix} a \\ b \end{bmatrix}_\omega = \binom{\lfloor \frac{a}{d} \rfloor}{\lfloor \frac{b}{d} \rfloor} \begin{bmatrix} a - d\lfloor \frac{a}{d} \rfloor \\ b - d\lfloor \frac{b}{d} \rfloor \end{bmatrix}_\omega .$$

In particular, if $b \equiv 0 \pmod d$

$$\begin{bmatrix} a \\ b \end{bmatrix}_\omega = \binom{\lfloor \frac{a}{d} \rfloor}{\lfloor \frac{b}{d} \rfloor}.$$

Proof. A proof may be found, for example, in [5, Theorem 2.2]. □

For greater clarity, we formulate the evaluation of the q-multinomial coefficient appearing in Theorem 1.8 as a lemma:

Lemma 3.2. *Let ω be a primitive d^{th} root of unity with $d \geq 2$ and $n \equiv 0 \pmod d$. Then*

$$\frac{1}{[n-1]_\omega} \begin{bmatrix} n-2+a+b+c \\ n-2, a, b, c \end{bmatrix}_\omega = \binom{\lfloor \frac{n+a+b+c}{d} \rfloor - 1}{\frac{n}{d} - 1, \lfloor \frac{a}{d} \rfloor, \lfloor \frac{b}{d} \rfloor, \lfloor \frac{c}{d} \rfloor}$$

if one of a, b and c equals $1 \pmod d$ and the others equal $0 \pmod d$, or $d = 2$ and $a \equiv b \equiv c \equiv 0 \pmod d$. Furthermore,

$$\frac{1}{[n-1]_\omega} \begin{bmatrix} n-2+a+b+c \\ n-2, a, b, c \end{bmatrix}_\omega = 0$$

otherwise, except if $d > 2$ and $a \equiv b \equiv c \equiv 0 \pmod d$ – we do not make a statement for this case.

Proof. Let us first write rewrite the multinomial coefficient as a product of binomial coefficients:

$$\frac{1}{[n-1]_q} \begin{bmatrix} n-2+a+b+c \\ n-2, a, b, c \end{bmatrix}_q$$
$$= \frac{1}{[n-1]_q} \begin{bmatrix} n-2+a \\ a \end{bmatrix}_q \begin{bmatrix} n-2+a+b \\ b \end{bmatrix}_q \begin{bmatrix} n-2+a+b+c \\ c \end{bmatrix}_q .$$

Since $\frac{1}{[n-1]_q} \begin{bmatrix} n-2+a+b+c \\ n-2,a,b,c \end{bmatrix}_q$ is symmetric in a, b and c it is sufficient to consider the following cases to prove the first equality:

If $a \equiv 1 \pmod{d}$, the q-Lucas theorem and $[n-1]_\omega = [d-1]_\omega$ implies

$$\begin{aligned}\frac{1}{[n-1]_\omega}\begin{bmatrix} n-2+a \\ a \end{bmatrix}_\omega &= \frac{1}{[d-1]_\omega}\binom{\lfloor \frac{n-2+a}{d} \rfloor}{\lfloor \frac{a}{d} \rfloor}\begin{bmatrix} d-1 \\ 1 \end{bmatrix}_\omega \\ &= \binom{\lfloor \frac{n-2+a}{d} \rfloor}{\lfloor \frac{a}{d} \rfloor} = \binom{\lfloor \frac{n+a}{d} \rfloor - 1}{\lfloor \frac{a}{d} \rfloor}.\end{aligned}$$

If $a \equiv 0 \pmod{d}$ and $d = 2$ we obtain by similar means

$$\begin{aligned}\frac{1}{[n-1]_\omega}\begin{bmatrix} n-2+a \\ a \end{bmatrix}_\omega &= \frac{1}{[1]_\omega}\binom{\frac{n-2+a}{d}}{\frac{a}{d}} \\ &= \binom{\lfloor \frac{n+a}{d} \rfloor - 1}{\lfloor \frac{a}{d} \rfloor}.\end{aligned}$$

Suppose now that $b \equiv c \equiv 0 \pmod{d}$ and $a \equiv 0 \pmod{d}$ or $a \equiv 1 \pmod{d}$. Then, again by the q-Lucas theorem,

$$\begin{bmatrix} n-2+a+b \\ b \end{bmatrix}_\omega = \binom{\lfloor \frac{n-2+a+b}{d} \rfloor}{\frac{b}{d}} = \binom{\lfloor \frac{n+a+b}{d} \rfloor - 1}{\lfloor \frac{b}{d} \rfloor}$$

and

$$\begin{bmatrix} n-2+a+b+c \\ c \end{bmatrix}_\omega = \binom{\lfloor \frac{n-2+a+b+c}{d} \rfloor}{\frac{c}{d}} = \binom{\lfloor \frac{n+a+b+c}{d} \rfloor - 1}{\lfloor \frac{c}{d} \rfloor}.$$

To show the second equality, again taking advantage of the symmetry, we only have to distinguish two cases: on the one hand, if $a \equiv b \equiv 1 \pmod{d}$, we have

$$\begin{bmatrix} n-2+a+b \\ b \end{bmatrix}_\omega = \binom{\frac{n-2+a+b}{d}}{\frac{b-1}{d}}\begin{bmatrix} 0 \\ 1 \end{bmatrix}_\omega = 0.$$

On the other hand, if $a \equiv e \pmod{d}$ with $e \geq 2$, then

$$\begin{bmatrix} n-2+a \\ a \end{bmatrix}_\omega = \binom{\lfloor \frac{n-2+a}{d} \rfloor}{\lfloor \frac{a}{d} \rfloor}\begin{bmatrix} e-2 \\ e \end{bmatrix}_\omega = 0.$$

□

Proof of Theorem 1.8. Let $d \geq 2$, $N+1 \equiv 0 \pmod{d}$ and ω a primitive d^{th} root of unity. We have to show that $P_{N,k,\ell,m}(\omega) = P^{(d)}_{N,k,\ell,m}$. To this end, let

$$M_q = \frac{1}{[N]_q}\begin{bmatrix} N-1+k+\ell+m \\ N-1, k, \ell, m \end{bmatrix}_q, \quad B_q = \begin{bmatrix} N-2-k-\ell-m \\ \ell+m-1 \end{bmatrix}_q,$$

$$M = \binom{\frac{N+1}{d} - 1 + \lfloor \frac{k+\ell+m}{d} \rfloor}{\frac{N+1}{d} - 1, \lfloor \frac{k}{d} \rfloor, \lfloor \frac{\ell}{d} \rfloor, \lfloor \frac{m}{d} \rfloor}, \qquad B = \binom{\lfloor \frac{N-2-k-\ell-m}{d} \rfloor}{\lfloor \frac{\ell+m-1}{d} \rfloor}.$$

Let us first check the cases where $M_\omega = M$:

(i) $d = 2$, $k \equiv \ell \equiv m \equiv 0 \pmod{d}$:
we have $N - 2 - k - \ell - m \equiv \ell + m - 1 \equiv 1 \pmod{d}$, thus the q-Lucas theorem entails $B_\omega = B$.

(ii) $k \equiv 1 \pmod{d}$ and $\ell \equiv m \equiv 0 \pmod{d}$:
we have $N - 2 - k - \ell - m \equiv -4 \pmod{d}$ and $\ell + m - 1 \equiv -1 \pmod{d}$. Thus,
(a) if $d = 2$, the q-binomial coefficient on the right hand side of the q-Lucas theorem is $\begin{bmatrix}0\\1\end{bmatrix}_\omega = 0$, and thus $B_\omega = 0$.
(b) If $d \geq 4$, it is $\begin{bmatrix}d-4\\d-1\end{bmatrix}_\omega = 0$, and therefore $B_\omega = 0$.
(c) However, if $d = 3$, it is $\begin{bmatrix}2d-4\\d-1\end{bmatrix}_\omega = \begin{bmatrix}2\\2\end{bmatrix}_\omega = 1$, and $B_\omega = B$.

(iii) $k \equiv 0 \pmod{d}$ and $\ell \equiv m - 1 \equiv 0 \pmod{d}$ or $\ell - 1 \equiv m \equiv 0 \pmod{d}$:
in this case $\ell + m - 1 \equiv 0 \pmod{d}$, and the q-Lucas theorem entails $B_\omega = B$.

It remains to check that $M_\omega \cdot B_\omega = 0$ in all other cases. Suppose that $M_\omega \neq 0$, then $d > 2$ and $k \equiv \ell \equiv m \equiv 0 \pmod{d}$. Thus $N - 2 - k - \ell - m \equiv -3 \pmod{d}$ and $\ell + m - 1 \equiv -1 \pmod{d}$, and the q-binomial coefficient on the right hand side of the q-Lucas theorem is $\begin{bmatrix}d-3\\d-1\end{bmatrix}_\omega = 0$. □

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References

[1] François Bergeron, Gilbert Labelle, and Pierre Leroux, *Combinatorial species and tree-like structures*, Encyclopedia of Mathematics and its Applications, vol. 67, Cambridge University Press, Cambridge, 1998, Translated from the 1994 French original by Margaret Readdy, With a foreword by Gian-Carlo Rota.

[2] Alan Guo, *Cyclic sieving phenomenon in non-crossing connected graphs*, Preprint (2010), math.CO/1007.4584.

[3] Thorsten Holm, Peter Jørgensen, and Martin Rubey, *Ptolemy diagrams and torsion pairs in the cluster category of Dynkin type A_n*, Preprint (2010), math.CO/1010.1184.

[4] Victor Reiner, Dennis Stanton, and Dennis White, *The cyclic sieving phenomenon*, Journal of Combinatorial Theory, Series A **108** (2004), no. 1, 17–50, 10.1016/j.jcta.2004.04.009.

[5] Bruce E. Sagan, *Congruence properties of q-analogs*, Advances in Mathematics **95** (1992), no. 1, 127–143, 10.1016/0001-8708(92)90046-N.

[6] Richard P. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999, With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

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